Seat No:______________ **Enrollment No:______________**

PARUL UNIVERSITY

FACULTY OF APPLIED SCIENCE

M.Sc., Summer 2017-18 Examination

Semester: 2 **Date: 14/05/2018**
Subject Code: 11206154 **Time: 10:30 AM to 01:00 PM**
Subject Name: Advanced Linear Algebra **Total Marks: 60**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate full marks.
3. Make suitable assumptions wherever necessary.
4. Start new question on new page.

Q.1. A) Answer the following Questions.(Any one) **(08)**

(a) Let $T: V \to W$ be a linear transformation. Then prove the following:
i) If X is a subspace of V, then $T(X)$ is a subspace of W.
ii) If Y is a subspace of W, then $T^{-1}(Y)$ is a subspace of V containing $Ker(T)$.
iii) Assume X_1, X_2 are subspaces of V both containing $Ker(T)$. If $T(X_1) = T(X_2)$, then $X_1 = X_2$.

(b) Let V be an n-dimensional vector space and T an operator on V . Then prove that there exists a vector z such that $\mu_T(x) = \mu_{T,z}(x)$.

Q.1. B) Answer the following questions. (Any two) **(08)**

(a) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(v) = \begin{pmatrix} 2 & -1 & 1 \\ -3 & 4 & -5 \\ -3 & 3 & -4 \end{pmatrix} v$. Let $v = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$. Determine minimal polynomial of T with respect to v.

(b) Let V be a vector space with basis $B_v = (v_1, v_2, \ldots, v_n)$, W be a vector space with basis $B_w = (w_1, w_2, \ldots, w_m)$, and $T \in L(V, W)$. Let $B_v{}' = (f_1, f_2, \ldots, f_n)$ be the basis dual to B_v and $B'_w = (g_1, g_2, \ldots, g_m)$ be the basis dual to B_w.Then show that $M_{T'}(B'_w, B'_v) = M_T(B_v, B_w)^{tr}$.

(c) Let V be an n-dimensional vector space and T an operator on V. Then Show that there exists a non-zero polynomial $f(x)$ of degree at most n^2 such that $f(T) = 0_{V \to V}$.

Q.2. A) Answer the following questions.(Any two) **(08)**

(a) Assume $T: V \to W$ is a linear transformation. Then show that T is injective if and only if $Ker(T) = \{0_V\}$.

(b) Let V be a finite-dimensional inner product space and assume that $f \in V'$. Then prove that there exists a unique vector $v \in V$ such that $f(u) = < u, v >$ for all $u \in V$.

(c) Let $w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ be two vectors in $\mathbb{R}^4$. Set $W = span(w_1, w_2)$. Then find orthogonal projection of the vector $v = \begin{pmatrix} 1 \\ 3 \\ -4 \\ 6 \end{pmatrix}$ onto W.

Q.2. B) Answer of the following questions.(Any one) **(06)**

(a) Let V be an inner product space and $u, v \in V$ be vectors in V. Then show that $||u + v|| \leq ||u|| + ||v||$. Moreover, if $u, v \neq 0$, equality holds only if there exists $\lambda > 0$ such that $v = \lambda u$.

(b) Let $f(x)$ and $d(x) \neq 0$ be polynomials with coefficients in F. Prove that there exists unique polynomials $q(x)$ and $r(x)$, which satisfy $f(x) = q(x)d(x) + r(x)$, where either $r(x) = 0$ or $deg(r(x)) < deg(d(x))$.

Q.3. A) Answer the following questions. **(08)**

(a) Let V be an inner product space and T an operator on V. Then prove that there exists an isometry S on V such that $T = S\sqrt{T^*T}$.

(b) Let V be a finite inner product space and T an operator on V. Suppose $S = \{v_1, v_2, \dots, v_n\}$ is an orthonormal basis of V such that $T(S)$ is an orthonormal basis of V. Then show that T is an isometry.

Q.3. B) Do as directed. **(08)**

(a) Let $V = \mathbb{R}_2[x]$ be a vector space over $\mathbb{R}$. Then find $||x^2 + 1||$, if the inner is product defined as $< f(x), g(x) > = \int_0^1 f(x)\overline{g(x)}$, where $f(x), g(x) \in \mathbb{R}_2[x]$.

(b) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$be a linear transformation defined by $T(x, y, z) = (x + y, y + z, 3z)$. Then find matrix of T, $M_T(B_1, B_2)$, where $B_1\{(1,0,0), (0,1,0), (0,0,1)\}$ and $B_2 = \{(1,0,-1), (0,1,-1), (0,0,1)\}$

(c) Let V be a vector space over $\mathbb{F}$ and $v \in V$. Then show that $0. v = 0$.

(d) Let V be an inner product space and let $u \in U$. Prove that $u^\perp$ is a subspace of V.

Q.4. A) State whether the following statements are true or false. Justify your answer. **(08)**

(a) Let $D: \mathbb{R}^3[x] \to \mathbb{R}^2[x]$ be a linear transformation defined by $D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$. Then D is one-one.

(b) If T be a normal operator on V. Then for all vectors $v \in V$, $||T\ (v)|| = ||T^*(v)||$.

(c) Let V be an inner product space and W be a subspace of V. Then $W \cap W^\perp \neq \{0\}$.

(d) Assume T is normal and λ is a scalar. Then $T - \lambda I$ is normal.

Q.4. B) Select the most appropriate answer for the following multiple choice questions. **(06)**

(1) Let T be a self-adjoint operator on V. Then eigenvalues of T are ______.

a) purely imaginary b) complex
c) real d) irrational

(2) Let W be a 3-dimensional subspace of $\mathbb{R}^7$ over $\mathbb{R}$. Then what is the dimension of $W^\perp$?

a) 3 b) 4
c) 7 d) 1

(3) If T is an isometry on a finite dimensional vector space V. Then which of the following correct?

a) T is one-one but not onto b)T is one-one and onto
c) T is neither one-one or onto d) T is onto but not one-one

(4) Let $T: V \to W$ be an isomorphism and $n = \dim(V)$, $m = \dim(W)$. Then which of the following must be true?

a) $m > n$ b) $m = n$
c) $n < m$ d) No relation between m and n.

(5) If T is an operator on V, then which of the following correct?

a) T^*T is normal b) TT^* is normal
c) Both T^*T and TT^* are normal d) neither T^*T nor TT^* is normal

(6) Consider the transformation $T: \mathbb{R}_2[x] \to \mathbb{R}^2$ given by $T(f) = \big(f(1), f(2)\big)$. Then $Ker(T)$= _______.

a) $span\{(x-1)(x-2)\}$ b) $span\{(x-1)\}$
c) $\{0\}$ d) $span\{(x-2)\}$