Seat No: ____________ Enrollment No: ____________

PARUL UNIVERSITY

FACULTY OF APPLIED SCIENCE

M.Sc., Summer 2018-19 Examination

Semester: 4 **Date: 01/04/2019**
Subject Code: 11206251 **Time: 02:00 pm to 04:30 pm**
Subject Name: Operator Theory **Total Marks: 60**

Instructions:
1. All questions are compulsory.
2. Figures to the right indicate full marks.
3. Make suitable assumptions wherever necessary.
4. Start new question on new page.

Q.1. A) Essay type/ Brief note (4x2) (Each of 04 marks) **(08)**

(a) Let X and Y be normed spaces and $T: X \rightarrow Y$ a linear operator. Then T is bounded if and only if it maps every bounded sequence (x_n) in X onto a sequence (Tx_n) in Y which has a convergent subsequence.

(b) Let $T: X \rightarrow X$ be a compact linear operator and $S: X \rightarrow X$ a bounded linear operator on a normed space X. Then prove that TS and ST are compact.

Q.1. B) Answer the following questions (Any two)

(a) If T_1 and $T_2$are compact linear operators from a normed space X into a normed space$Y$, show that $T_1 + T_2$ and T_2T_1 are compact linear operators. **(04)**

(b) Let $T: X \rightarrow X$ be a compact linear operator on a normed space X. Then for every $\lambda \neq 0$ the null space$N(T_\lambda)$ of $T_\lambda = T - \lambda I$ is finite dimensional. **(04)**

(c) Let B be a subset of a metric space X, then if B is relatively compact then show that it is totally bounded. **(04)**

Q.2. A) Answer the following questions.

(a) Do as Directed. (Each of 02 marks) **(04)**

1. Show that the eigen values of a Hermitian matrix are real.
2. Define holomorphy and local holomorphy.

(b) Let $T \in B(X, X)$, where X is a Banach space. If $||T||<1$, then $(1 - T)^{-1}$ exists as a bounded linear operator on the whole space X and find $(1 - T)^{-1}$ **(04)**

Q.2. B) Answer the following questions (Any two)

(a) State and prove the Hilbert relation. **(03)**

(b) Show that for a given linear operator T, the sets $\rho(T), \sigma_p(T), \sigma_c(T)$ and $\sigma_r(T)$ are mutually disjoint and their union is the complex plane. **(03)**

(c) Let $X = C[0,1]$ and define $T: X \rightarrow X$ by $Tx = vx$, where $v \in X$ is fixed. Find $\sigma(T)$. Note that $\sigma(T)$ is closed. **(03)**

Q.3. A) Essay type/ Brief note (4x2) (Each of 04 marks) **(08)**

(a) Prove that the spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T: H \rightarrow H$ on a complex Hilbert space H is real.

(b) Prove that the residual spectrum $\sigma_r(T)$ of a bounded self adjoint linear operator $T: H \rightarrow H$ on a complex Hilbert space H is empty.

Q.3. B) Answer the following questions (Any two)

(a) Do as Directed. (Each of 02 marks) **(04)**

1. What are positive operators?
2. Under what conditions will the projection P be $(i) P = 0$? $(ii) P = I$?

(b) Let S and T be bounded self-adjoint linear operators on a complex Hilbert space. If $S \leq T$ and $S \geq T$, show that $S = T$. **(04)**

(c) State and prove the Hellinger-Toeplitz theorem. **(04)**

Q.4. A) Answer the following questions.

(a) Do as Directed. (Each of 02 marks) **(04)**

1. Define symmetric linear operator.
2. Explain closed linear operator.

(b) Let P_1 and P_2 be two projections on a Hilbert space H, then show that the sum $P = P_1 + P_2$ is a projection on H if and only if $Y_1 = P_1(H)$ and $P_2(H)$ are orthogonal. **(04)**

Q.4. B) Answer the following questions (Any two)

(a) A densely defined linear operator T in a complex Hilbert space H is symmetric if and only if $T \subset T^*$ **(03)**

(b) Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Then prove that all the eigen values of T are real. **(03)**

(c) Prove that the Hilbert adjoint operator $T^*$of a linear operator T is linear. **(03)**