**Seat No:** ______________ **Enrollment No:** ______________

# PARUL UNIVERSITY

## FACULTY OF APPLIED SCIENCE

## M.Sc. Summer 2018-19 Examination

**Semester: 2** **Date: 08/04/2019**
**Subject Code: 11206154** **Time: 10:30am to 1.00 pm**
**Subject Name: Advanced Linear Algebra** **Total Marks: 60**

**Instructions:**
1. All questions are compulsory.
2. Figures to the right indicate full marks.
3. Make suitable assumptions wherever necessary.
4. Start new question on new page.

**Q.1. A) Answer the following Questions.(Any one)** **(08)**

(a) Let $T: V \to W$ be a linear transformation. Then prove the following:
i) If $X$ is a subspace of $V$, then $T(X)$ is a subspace of $W$.
ii) If $Y$ is a subspace of $W$, then $T^{-1}(Y)$ is a subspace of $V$ containing $Ker(T)$.
iii) Assume $X_1, X_2$ are subspaces of $V$ both containing $Ker(T)$. If $T(X_1) = T(X_2)$, then $X_1 = X_2$.

(b) Let $V$ be an $n$-dimensional vector space and $T$ an operator on $V$ . Then prove that there exists a vector $z$ such that $\mu_T(x) = \mu_{T,z}(x)$.

**Q.1. B) Answer the following questions. (Any two)** **(08)**

(a) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(v) = \begin{pmatrix} 2 & -1 & 1 \\ -3 & 4 & -5 \\ -3 & 3 & -4 \end{pmatrix} v$. Let $v = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$. Determine minimal polynomial of $T$ with respect to $v$.

(b) Let $V$ be a vector space with basis $B_v = (v_1, v_2, \dots, v_n), W$ be a vector space with basis $B_w = (w_1, w_2, \dots, w_m)$, and $T \in L(V, W)$. Let $B_v{}' = (f_1, f_2, \dots, f_n)$ be the basis dual to $B_v$ and $B'_w = (g_1, g_2, \dots, g_m)$ be the basis dual to $B_w$.Then show that $M_{T'}(B'_w, B'_v) = M_T(B_v, B_w)^{tr}$.

(c) Let $V$ be an $n$-dimensional vector space and $T$ an operator on $V$. Then Show that there exists a non-zero polynomial $f(x)$ of degree at most $n^2$ such that $f(T) = 0_{V \to V}$ .

**Q.2. A) Answer the following questions.(Any two)** **(08)**

(a) Assume $T: V \to W$ is a linear transformation. Then show that $T$ is injective if and only if $Ker(T) = \{0_V\}$.

(b) Let $V$ be a finite-dimensional inner product space and assume that $f \in V'$. Then prove that there exists a unique vector $v \in V$ such that $f(u) = < u, v >$ for all $u \in V$.

(c) Let $w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ be two vectors in $\mathbb{R}^4$. Set $W = span(w_1, w_2)$. Then find orthogonal projection of the vector $v = \begin{pmatrix} 1 \\ 3 \\ -4 \\ 6 \end{pmatrix}$ onto $W$.

**Q.2. B) Answer of the following questions.(Any one)** **(06)**

(a) Let $V$ be an inner product space and $u, v \in V$ be vectors in $V$. Then show that $||u + v|| \le ||u|| + ||v||$. Moreover, if $u, v \ne 0$, equality holds only if there exists $\lambda > 0$ such that $v = \lambda u$.

(b) Let $f(x)$ and $d(x) \ne 0$ be polynomials with coefficients in F. Prove that there exists unique polynomials $q(x)$ and $r(x)$, which satisfy $f(x) = q(x)d(x) + r(x)$, where either $r(x) = 0$ or $deg(r(x)) < deg(d(x))$.

**(08)**

**Q.3. A) Answer the following questions.**

(a) Let $V$ be an inner product space and $T$ an operator on $V$. Then prove that there exists an isometry $S$ on $V$ such that $T = S\sqrt{T^*T}$.

(b) Let $V$ be a finite inner product space and $T$ an operator on $V$. Suppose $S = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis of $V$ such that $T(S)$ is an orthonormal basis of $V$. Then show that $T$ is an isometry.

**Q.3. B) Do as directed.** **(08)**

(a) Let $V = \mathbb{R}_2[x]$ be a vector space over $\mathbb{R}$. Then find $\left|\left|x^2 + 1\right|\right|$, if the inner product is defined as $< f(x), g(x) > = \int_0^1 f(x)\overline{g(x)}$, where $f(x), g(x) \in \mathbb{R}_2[x]$.

(b) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$be a linear transformation defined by $T(x, y, z) = (x + y, y + z, 3z)$. Then find matrix of $T$, $M_T(B_1, B_2)$, where $B_1\{(1,0,0), (0,1,0), (0,0,1)\}$ and $B_2 = \{(1,0,-1), (0,1,-1), (0,0,1)\}$

(c) Let $V$ be a vector space over $\mathbb{F}$ and $v \in V$. Then show that $0.v = 0$.

(d) Let $V$ be an inner product space and let $u \in U$. Prove that $u^\perp$ is a subspace of $V$.

**Q.4. A) State whether the following statements are true or false. Justify your answer.** **(08)**

(a) Let $D: \mathbb{R}^3[x] \to \mathbb{R}^2[x]$ be a linear transformation defined by $D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$. Then $D$ is one-one.

(b) If $T$ be a normal operator on $V$. Then for all vectors $v \in V$, $\left|\left|T\ (v)\right|\right| = \left|\left|T^*(v)\right|\right|$.

(c) Let $V$ be an inner product space and $W$ be a subspace of $V$. Then $W \cap W^\perp \neq \{0\}$.

(d) Assume $T$ is normal and $\lambda$ is a scalar. Then $T - \lambda I$ is normal.

**Q.4. B) Select the most appropriate answer for the following multiple choice questions.** **(06)**

(1) If $T$ is an isometry on a finite dimensional vector space $V$. Then which of the following correct?

a) T is one-one but not onto b)T is one-one and onto
c) T is neither one-one or onto d) T is onto but not one-one

(2) Let $W$ be a 3-dimensional subspace of $\mathbb{R}^7$ over $\mathbb{R}$. Then what is the dimension of $W^\perp$?

a) 3 b) 4
c) 7 d) 1

(3) Let $T: V \to W$ be an isomorphism and $n = \dim(V)$, $m = \dim(W)$. Then which of the following must be true?

a) $m > n$ b) $m = n$
c) $n < m$ d) No relation between $m$ and $n$.

(4) If $T$ is an operator on $V$, then which of the following correct?

a) $T^*T$ is normal b) $TT^*$ is normal
c) Both $T^*T$ and $TT^*$ are normal d) neither $T^*T$ nor $TT^*$ is normal

(5) Consider the transformation $T: \mathbb{R}_2[x] \to \mathbb{R}^2$ given by $T(f) = \big(f(1), f(2)\big)$. Then $Ker(T)$= _______.

a) $span\{(x-1)(x-2)\}$ b) $span\{(x-1)\}$
c) $\{0\}$ d) $span\{(x-2)\}$

(6) Let $T$ be a self-adjoint operator on V. Then eigenvalues of $T$ are ______.

a) purely imaginary b) complex
c) real d) irrational