Seat No:______________ Enrollment No:______________

PARUL UNIVERSITY

FACULTY OF APPLIED SCIENCE

M.Sc., Summer 2022-23 Examination

Semester: 4 **Date: 20-03-2023**
Subject Code: 11206251 **Time:** 2:00pm to 4:30pm
Subject Name: Operator Theory **Total Marks: 60**

Instructions:
1. All questions are compulsory.
2. Figures to the right indicate full marks.
3. Make suitable assumptions wherever necessary.
4. Start new question on new page.

Q.1. A) Essay type/ Brief note (4x2) (Each of 04 marks) **(08)**

(a) Prove that all matrices representing a given linear operator $T: X \to X$ on a finite dimensional normed space X relative to various bases for X have the same eigen values.

(b) The resolvent set $\rho(T)$ of a bounded linear operator T on a complex banach space X is open; hence the spectrum $\sigma(T)$ is closed.

Q.1. B) Answer the following questions (Any two)

(a) Short note/ Brief note (2x2)/ Schematically label the figures (2x2) (Each of 02 marks) **(04)**

1. Let $X = C[0,1]$ and define $T: X \to X$ by $Tx = vx$, where $v \in X$ is fixed. Find $\sigma(T)$. Note that $\sigma(T)$ is closed.
2. Define Point spectrum and continuous spectrum.

(b) State and prove the Hilbert relation. **(04)**

(c) Let $T \in B(X, X)$, where X is a Banach space. If $||T||<1$, then $(1-T)^{-1}$ exists as a bounded linear operator on the whole space X and find $(1-T)^{-1}$ **(04)**

Q.2. A) Answer the following questions.

(a) Short note/ Brief note (2x2)/ Fill in the blanks. (Each of 02 marks) **(04)**

1. Define $\epsilon - net$ and total boundedness.
2. State Compactness Criterion.

(b)Let $T: X \to X$ be a compact linear operator and $S: X \to X$ a bounded linear operator on a normed space X. Then prove that TS and ST are compact. **(04)**

Q.2. B) Answer the following questions (Any two)

(a) If T_1 and $T_2$are compact linear operators from a normed space X into a normed space$Y$, show that $T_1 + T_2$ and $T_2 T_1$ are compact linear operators. **(03)**

(b) For the identity operator I on a normed space X, find the eigen values and eigen spaces as well as $\sigma(I)$ and $R_\lambda(I)$. **(03)**

(c) Let B be a subset of a metric space X, then if B is relatively compact then show that it is totally bounded. **(03)**

Q.3. A) Essay type/ Brief note (4x2) (Each of 04 marks) **(08)**

(a)A bounded linear operator $P: H \to H$ on a Hilbert space H is a projection if and only if P is self-adjoint and idempotent.

(b)Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H. Then prove that the eigenvectors corresponding to different eigenvalues of T are orthogonal.

Q.3. B) Answer the following questions (Any two)

(a) Short note/ Brief note (2x2)/ Schematically label the figures (2x2) (Each of 02 marks) **(04)**

1. What are positive operators?
2. Under what conditions will the projection P be $(i) P = 0$? $(ii) P = I$?

(b) $P = P_1 P_2$ is a projection on H if and only if the projections P_1 and P_2 commute, that is $P_1 P_2 = P_2 P_1$ **(04)**

(c) State and prove the Hellinger-Toeplitz theorem. **(04)**

Q.4. A) Answer the following questions.

(a) Short note/ Brief note (2x2)/ Fill in the blanks. (Each of 02 marks) **(04)**

1.Define self-adjoint linear operator.

2.Explain closed linear operator.

(b) Let S and T be bounded self-adjoint linear operators on a complex Hilbert space. If $S \leq T$ and $S \geq T$, show that $S = T$. **(04)**

Q.4. B) Answer the following questions (Any two)

(a) Prove that the Hilbert adjoint operator $T^*$of a linear operator T is linear. **(03)**

(b) Prove that the residual spectrum $\sigma_r(T)$ of a bounded self adjoint linear operator $T: H \to H$ on a complex Hilbert space H is empty. **(03)**

(c) A densely defined linear operator T in a complex Hilbert space H is symmetric if and only if $T \subset T^*$ **(03)**